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A NOTEON STRONG $\Sigma^* - \mathcal{H}$ −OPENSETS IN GTS VIA HEREDITARY CLASS

R. Shankar
Department Of Mathematics,
Dwaraka Doss Goverdhan Doss Vaishnav College,
Chennai, Tamil Nadu, India.

Abstract - **The aim of this paper is to introduce and study the strong$\sigma^* - \mathcal{H}$ −open set in a generalized topological space (X, λ) with a hereditary class $\mathcal{H}$. We give discuss and some of their properties strong$\sigma^* - \mathcal{H}$ −open set and strong$\pi^* - \mathcal{H} -$ open set in a generalized topological space (X, λ) with a hereditary class $\mathcal{H}$. Also discuss various characterization strong$\sigma^* - \mathcal{H}$ −open set in a generalized topological space (X, λ) with a hereditary class $\mathcal{H}$ are given and its properties of such sets are discussed. Also we establish state some characterize of strong $\pi^* - \mathcal{H}$ −open set in a generalized topological space (X, λ) with a hereditary class $\mathcal{H}$.**

Keywords **-Hereditary class,$\pi^* - \mathcal{H}$ −open set, λ −codense, $t - \mathcal{H}$ −set,$\sigma^* - \mathcal{H}$ −open set, strong$\beta - \mathcal{H}$ −open set and $\mathcal{H}$ −open set.**
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I. INTRODUCTION

Let X be a nonempty set and $\wp(X)$ denotes the power set of X. Then the collection $\lambda \subseteq \wp(X)$is called a generalized topology (GT) on X [1] if $\emptyset \in \lambda$ and arbitrary union of members of λ belongs toλ. A set X, with a GT on it is said to be generalized topological space (GTS) (X, λ). Members of λ are called λ −open sets.A subset $E \subset X$ is said to be λ −closed sets if $X - E$ isλ −open set. For each $E \subset X$, we denote by$\text{int}_\lambda(E)$ [2] the union of all λ −open sets contained in E and by $\text{cl}_\lambda(E)$ [2] the intersection of all λ-closed sets containing E. A GTS $(X, \lambda)$is said to be a quasitopological space [4] (QTS) if $K, L \in \lambda$ implies $K \cap L \in \lambda$.A hereditary class (HC) $\mathcal{H}$ is a nonempty family of subset of X such that $L \subset M$, $M \in \mathcal{H}$ implies $L \in \mathcal{H}$ [2]. Hereditary class $\mathcal{H}$ is said to be λ −codense [2] if $\lambda \cap \mathcal{H} = \{\emptyset\}$ [2].For each $E \subset X$, a subset $E^*(\mathcal{H})$ is defined by $E^* = \{x \in X | N \cap E \notin \mathcal{H}$ for every $N \in \lambda$ such that $x \in N\}$ [3].If $\text{cl}^*_\lambda(E) = E \cup E^*$for each $E \subset X$, with respect to λ and a HC $\mathcal{H}$ of subsets of X, then $\lambda^* = \{E \subset X / \text{cl}^*_\lambda(X - E) = X - E\}$ [3] is a GT. Members of λ^* are called λ^* −open sets and its complement is called a λ^* −closed set. We denote the interior of E in (X, λ^*) is$\text{int}^*_\lambda(E)$ .A subset $E \subset X$ is said to be λ^* −dense (resp. λ −dense)if$\text{cl}^*_\lambda(E) = X$ (resp.$\text{cl}_\lambda(E) = X$).

Literature Review

In1997, Prof.Csaszar [1]nicely pointedout that the family of all open sets and all weak forms of open sets in atopological space (X, τ) can be obtained from monotonic functions defined on $\wp(X)$, the family of all subsets of X. His further study of generalized open sets using monotonic functionson$\wp(X)$. The collection of all monotonic function from $\wp(X)$ to $\wp(X)$ where X is any non-empty set and denote it by$\Gamma(X)$. For $\gamma \in \Gamma, \lambda = \{E \subset X | E \subset \gamma(E)\}$.In2002,Csaszar,definedgeneralized topology [2]. He proved that $\emptyset \in \lambda$ and λ is closed under arbitrary union. He named such families as Generalized Topology and the pair (X, λ) is called generalized topological space. Elements of λ are called λ −open sets and the complement of an λ −open set is called a $\lambda -$ closed set. The union of all λopen sets contained in a subset E of X is denoted by $\text{int}_\lambda(E)$ and is called the λ −interior of E. The intersection of all λ −closed sets containing E is called the λ −closure of E and is denoted by $\text{cl}_\lambda(E)$. In 2005, Csaszar [2] defined the generalized open sets and discuss their relationship for a generalized topology λ on a nonempty set X, the generalized open sets $\lambda - \alpha$ −open, $\lambda - \sigma$ −open, $\lambda - \pi$ −open and $\lambda - \beta$ −open sets. In 2006, Csaszar defined the normal generalized topological space.In2007,Csaszar [3]definedanonemptyclass of subsets ofanonempty set X, called as hereditary class $\mathcal{H}$ and studied modification of generalized topology with hereditary classes. He defined the concepts codense and strongly completely codense hereditary classes. Also he defined about quazi topology, which is closed under finite intersection. In 2009, Csaszar defined and studied the product of generalized topologies. Also, generalized topology with hereditary classes are studied by Y.K. Kim and W.K. Min [21], V. Renukadevi and Sheena Scaria [5], V. Renukadevi and K. Karuppayi [6].

II. MATERIALS AND METHODS

Lemma 2.1 [1] Let (X, λ) be a GTSand $E, F \subset X$. Then the following hold.

(i) $\text{int}_\lambda(\emptyset) = \emptyset$.
(ii) $\text{int}_\lambda(X) = X$.
(iii) $\text{int}_\lambda(\text{int}_\lambda(E)) = \text{int}_\lambda(E)$.
(iv) $E \in \lambda \Leftrightarrow \text{int}_\lambda(E) = E$.
(v) $\text{int}_\lambda(E \cap F) = \text{int}_\lambda(E) \cap \text{int}_\lambda(F)$.

(vi) $\text{int}_\lambda(E) \cup \text{int}_\lambda(F) \subset \text{int}_\lambda(E \cup F)$.

Lemma 2.2 [1] Let (X, λ) be a GTS with a hereditary class $\mathcal{H}$ and $E, F \subset X$. Then the following hold.
(i) $\text{cl}_\lambda(\emptyset) = \emptyset$
(ii) $\text{cl}_\lambda(X) = X$.
(iii) E is λ −closed $\Leftrightarrow \text{cl}_\lambda(E) = E$.
(iv) $\text{cl}_\lambda(E \cap F) \subset \text{cl}_\lambda(E) \cap \text{cl}_\lambda(F)$.
(v) $\text{cl}_\lambda(E \cup F) = \text{cl}_\lambda(E) \cup \text{cl}_\lambda(F)$
(vi) $\text{cl}_\lambda(\text{cl}_\lambda(E)) = \text{cl}_\lambda(E)$.

Lemma. 2.3. [3] Let (X, λ) be a GTS with a HC $\mathcal{H}$ and $S, T \subset X$. Then the following hold.
(i) $S \subset T$ implies $S^* \subset T^*$.
(ii) $(S^*)^* = S^*$ for every $S \subset X$.
(iii) $S \subset T \subset X$ implies $\text{cl}^*_\lambda(S) \subset \text{cl}^*_\lambda(T)$.
(iv) $(E \cup E^*)^* \subset E^*$ for every $E \subset X$.
(v) $(S \cup T)^* = S^* \cup T^*$.
(vi) $\lambda \subset \lambda^*$.
(vii) F is λ^* − closed if and only if $F^* \subset F$.
(viii) $\beta = \{N - K : N \in \lambda, K \in \mathcal{H}\}$ is a base for λ^*.

Lemma 2.4. [6] Let (X, λ) be a GTS with a HC $\mathcal{H}$ and $E \subset X$. Then the following hold.
(i) If $N \in \lambda$, then $N \cap \text{cl}^*_\lambda(E) \subset \text{cl}^*_\lambda(N \cap E)$.
(ii) If $N \in \lambda$, then $N \cap E^* \subset (N \cap E)^*$.

Lemma 2.5.[5] Let (X, λ) be a GTS with a λ −codense hereditary class $\mathcal{H}$ and $E \subset X$. If $E \subset E^*$, then $E^* = \text{cl}_\lambda(E) = \text{cl}^*_\lambda(E) = \text{cl}_\lambda(E^*)$.

Lemma 2.6.[3] Let (X, λ) be a GTS with a hereditary class $\mathcal{H}$ and $E \subset X$. If E is a λ −semi closed set, then $\text{int}_\lambda(E) = \text{int}_\lambda(\text{cl}_\lambda(E))$.

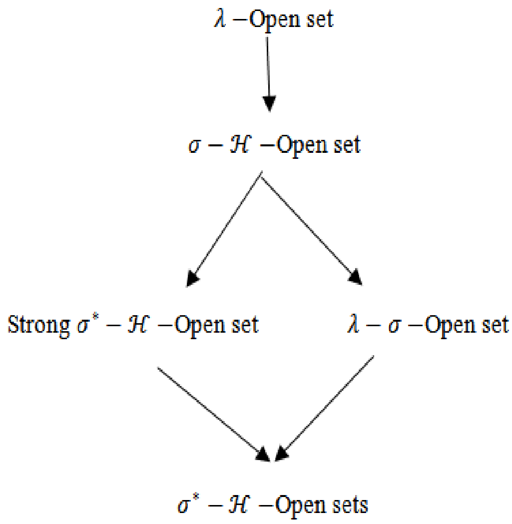
Lemma 2.7. [21] Let (X, λ) be a GTS with a hereditary class $\mathcal{H}$ and $E \subset X$. Then the following hold.
(a) $\text{int}_\lambda(E) \subset \text{int}^*_\lambda(E) \subset E$.
(b) $\text{int}^*_\lambda(E) = X - \text{cl}^*_\lambda(X - E)$.
(c) $\text{cl}^*_\lambda(E) = X - \text{int}^*_\lambda(X - E)$.
(d) $E \subset \text{cl}^*_\lambda(E) \subset \text{cl}_\lambda(E)$.

Lemma 2.8.[21] Let (X, λ) be a GTS with a λ −codense hereditary class $\mathcal{H}$ and $E \subset X$. Then the following hold.
(i) $\text{int}_\lambda(E) = \text{int}^*_\lambda(E)$, for every λ^* −closed set E.
(ii) $\text{cl}_\lambda(E) = \text{cl}^*_\lambda(E)$, for every λ^* −open set E.

Definition 2.9. A subset $E \subset X$ of a GTS (X, λ) with a hereditary class $\mathcal{H}$ is said to be
1. λ^* −Dense [3] in itself if $E \subset E^*$.
2. λ^* −Perfect [3] if $E = E^*$.
3. λ^* −Closed [3] if $E^* \subset E$.
4. $\mathcal{H}$ −open [3] if $E \subset \text{int}_\lambda(E^*)$
5. $\sigma - \mathcal{H}$ −open [3] if $E \subset \text{cl}^*_\lambda(\text{int}_\lambda(E))$.
6. Weakly semi−$\mathcal{H}$ −open [11] if $E \subset \text{cl}^*_\lambda\left(\text{int}_\lambda(\text{cl}_\lambda(E))\right)$.
7. Almost strong−$\mathcal{H}$ −open [11] if $E \subset \text{cl}^*_\lambda(\text{int}_\lambda(E^*))$.
8. $\sigma^* - \mathcal{H}$ −open [6]if $E \subset \text{cl}_\lambda(\text{int}^*_\lambda(E))$.
9. $\pi^* - \mathcal{H}$ −open [6] if $E \subset \text{int}^*_\lambda(\text{cl}_\lambda(E))$.
10. $\beta^* - \mathcal{H}$ −open [6] if $E \subset \text{cl}_\lambda\left(\text{int}^*_\lambda(\text{cl}_\lambda(E))\right)$.
11. $\pi - \mathcal{H}$ −open [3] if $E \subset \text{int}_\lambda(\text{cl}^*_\lambda(E))$.
12. $\alpha - \mathcal{H}$ −open [3] if $E \subset \text{int}_\lambda\left(\text{cl}^*_\lambda(\text{int}_\lambda(E))\right)$.
13. $\mathcal{H}$R −closed set [12]$E = \text{cl}^*_\lambda(\text{int}_\lambda(E))$.
14. $\alpha^* - \mathcal{H}$ −open [16] if $\text{int}_\lambda\left(\text{cl}^*_\lambda(\text{int}_\lambda(E))\right) = \text{int}_\lambda(E)$.
15. $t - \mathcal{H}$ −set [7] if $\text{int}_\lambda(E) = \text{int}_\lambda(\text{cl}^*_\lambda(E))$.
16. Strong $\beta - \mathcal{H}$ −open [13] if $E \subset \text{cl}^*_\lambda(\text{int}_\lambda(\text{cl}^*_\lambda(E))$.
17. Strong $\pi^* - \mathcal{H}$ −open [22] set if $\text{int}^*_\lambda(\text{cl}^*_\lambda(E))$.

The complement of a strong $\beta - \mathcal{H}$ − open (resp. $\sigma - \mathcal{H}$ −open, $\sigma^* - \mathcal{H}$ −open, $\pi^* - \mathcal{H}$ −open, $\beta^* - \mathcal{H}$ −open, $\alpha^* - \mathcal{H}$ −open, $\pi - \mathcal{H}$ −open, weakly semi−$\mathcal{H}$ −open, Strong $\pi^* - \mathcal{H}$ −open, $\alpha - \mathcal{H}$ −open) set is said to a strong $\beta - \mathcal{H}$ −closed (resp. $\sigma - \mathcal{H}$ − Closed, $\sigma^* - \mathcal{H}$ −closed, $\pi^* - \mathcal{H}$ −closed, $\beta^* - \mathcal{H}$ −closed, $\alpha^* - \mathcal{H}$ −closed, $\pi - \mathcal{H}$ −closed, weakly semi−$\mathcal{H}$ −closed, Strong $\pi^* - \mathcal{H}$ −closed, $\pi - \mathcal{H}$ −closed, $\alpha - \mathcal{H}$ −closed) set.

III. RESULTS AND DISCUSSION



3.1 Strong$\sigma^* - \mathcal{H}$ −open Sets

Definition 3.1. A subset $E \subset X$ of a GTS (X, λ) with a hereditary class $\mathcal{H}$ is said to be a strong $\sigma^* - \mathcal{H}$ −open set if $E \subset \text{cl}^*_\lambda(\text{int}^*_\lambda(E))$. The family of all strong $\sigma^* - \mathcal{H}$ −open set is denoted by $S\sigma^*\mathcal{H}(X, \lambda)$.If its compliment is strong $\sigma^* - \mathcal{H}$ −closed set.

Theorem 3.2. Let (X,λ) be a GTS with a hereditary class $\mathcal{H}$ and $E \subset X$. If every $\sigma-\mathcal{H}$ −open set is a strong $\sigma^*-\mathcal{H}$ −open set.

Proof. Let E be a $\sigma-\mathcal{H}$ −open set. Then $E \subset cl^*_\lambda(int_\lambda(E))$. Since λ^* finer than λ and so $E \subset cl^*_\lambda(int_\lambda(E)) \subset cl^*_\lambda(int^*_\lambda(E))$. Therefore, E is a strong $\sigma^*-\mathcal{H}$ −open set.

The following Example 3.3 shows that the converse of Theorem 3.2 is not true.

Example 3.3. Let $X=\{\chi_1,\chi_2,\chi_3,\chi_4\}, \lambda = \{\emptyset, X, \{\chi_1,\}, \{\chi_1,\chi_2\}, \{\chi_3,\chi_4\}, \{\chi_1,\chi_3,\chi_4\}\}$ and $\mathcal{H} = \{\emptyset, \{\chi_1\}, \{\chi_4\}, \{\chi_1,\chi_4\}\}$. If $E=\{\chi_3\}$ then $E^* = \{\chi_2,\chi_3,\chi_4\}$. Now $cl^*_\lambda(int^*_\lambda(E)) = cl^*_\lambda(int^*_\lambda(\{\chi_3\})) = cl^*_\lambda(X - cl^*_\lambda(X-\{\chi_3\})) = cl^*_\lambda(X - cl^*_\lambda(\{\chi_1,\chi_2,\chi_4\})) = cl^*_\lambda(X - (\{\chi_1,\chi_2,\chi_4\} \cup \{\chi_2\})) = cl^*_\lambda(X-\{\chi_1,\chi_2,\chi_4\}) = cl^*_\lambda(\{\chi_3\}) = \{\chi_3\} \cup \{\chi_2,\chi_3,\chi_4\} = \{\chi_2,\chi_3,\chi_4\} \supset E$. Therefore E is a strong $\sigma^*-\mathcal{H}$ −open set. But $cl^*_\lambda(int_\lambda(E)) = cl^*_\lambda(\emptyset) = \emptyset \not\supset E$. Hence E is not a $\sigma-\mathcal{H}$ −open set.

Theorem 3.4. Let (X,λ) be a GTS with a hereditary class $\mathcal{H}$ and $E \subset X$. If every strong $\sigma^*-\mathcal{H}$ −open set is a $\sigma^*-\mathcal{H}$ −open set.

Proof. Let E be a strong $\sigma^*-\mathcal{H}$ −open set. Then $E \subset cl^*_\lambda(int^*_\lambda(E))$. Since λ^* finer than λ and so $E \subset cl^*_\lambda(int^*_\lambda(E)) \subset cl_\lambda(int^*_\lambda(E))$. Therefore, E is a $\sigma^*-\mathcal{H}$ −open set.

The following Example 3.5 shows that the converse of Theorem 3.4 is not true.

Example 3.5. Let $X=\{\chi_1,\chi_2,\chi_3,\chi_4\}, \lambda = \{\emptyset, X, \{\chi_1,\}, \{\chi_2\}, \{\chi_1,\chi_2\}\}$ and $\mathcal{H} = \{\emptyset, \{\chi_2\}, \{\chi_3\}, \{\chi_2,\chi_3\}\}$. If $E = \{\chi_2,\chi_3\}$ then $E^* = \emptyset$. Now, $cl_\lambda(int^*_\lambda(E)) = cl_\lambda(int^*_\lambda(\{\chi_2,\chi_3\})) = cl_\lambda(X - cl^*_\lambda(X - \{\chi_2,\chi_3\})) = cl_\lambda(X - cl^*_\lambda(\{\chi_1,\chi_4\})) = cl_\lambda(X - (\{\chi_1,\chi_4\} \cup \{\chi_1,\chi_3,\chi_4\})) = cl_\lambda(X - cl^*_\lambda(\{\chi_1,\chi_4\})) = cl_\lambda(X - (\{\chi_1,\chi_4\} \cup \{\chi_1,\chi_3,\chi_4\})) = cl_\lambda(X - \{\chi_1,\chi_3,\chi_4\}) = cl_\lambda(\{\chi_2\}) = \{\chi_2,\chi_3,\chi_4\} \supset E$. Therefore E is a $\sigma^*-\mathcal{H}$ −open set. But $cl^*_\lambda(int^*_\lambda(E)) = cl^*_\lambda(int^*_\lambda(\{\chi_2,\chi_3\})) = cl^*_\lambda(X - cl^*_\lambda(X-\{\chi_2,\chi_3\})) = cl^*_\lambda(X - cl^*_\lambda(\{\chi_1,\chi_4\})) = cl^*_\lambda(X - \{\chi_1,\chi_3,\chi_4\}) = cl^*_\lambda(\{\chi_2\}) = \{\chi_2\} \cup \emptyset = \{\chi_2\} \not\supset E$. Hence E is not a strong $\sigma^*-\mathcal{H}$ −open set.

Theorem 3.6. Let (X,λ) be a GTS with a hereditary class $\mathcal{H}$ and $E \subset X$. Then E is a strong $\sigma^*-\mathcal{H}$ −open set $\Leftrightarrow$ $cl^*_\lambda(E) = cl^*_\lambda(int^*_\lambda(E))$.

Proof. Let E be a strong $\sigma^*-\mathcal{H}$ −open set. Then $E \subset cl^*_\lambda(int^*_\lambda(E))$. By Lemma, $cl^*_\lambda(E) \subset cl^*_\lambda\left(cl^*_\lambda(int^*_\lambda(E))\right) = cl^*_\lambda(int^*_\lambda(E)) \subset cl^*_\lambda(E)$. Hence $cl^*_\lambda(E) = cl^*_\lambda(int^*_\lambda(E))$. Conversely, let $cl^*_\lambda(E) = cl^*_\lambda(int^*_\lambda(E))$, since $E \subset cl^*_\lambda(E) \Longrightarrow E \subset cl^*_\lambda(int^*_\lambda(E))$.

Theorem 3.7. Let (X,λ) be a GTS with a hereditary class $\mathcal{H}$ and $E \subset X$. If E is a $\mathcal{H}R$ −closed set then, E is a strong $\sigma^*-\mathcal{H}$ −open set.

Proof. Let E be a $\mathcal{H}R$ −closed set. Then $E = cl^*_\lambda(int_\lambda(E)) \subset cl^*_\lambda(int^*_\lambda(E)) \Longrightarrow$ E is a strong $\sigma^*-\mathcal{H}$ −open set.

The following Example 3.8 shows that the converse of Theorem 3.7 is not true.

Example 3.8. Let $X=\{\chi_1,\chi_2,\chi_3,\chi_4\}, \lambda = \{\emptyset, X, \{\chi_4\}, \{\chi_1,\chi_3\}, \{\chi_1,\chi_3,\chi_4\}\}$ and $\mathcal{H} = \{\emptyset, \{\chi_3\}, \{\chi_4\}, \{\chi_3,\chi_4\}\}$. If $E = \{\chi_1\}$. Now $cl^*_\lambda(int^*_\lambda(E)) = cl^*_\lambda(X - cl^*_\lambda(X-E)) = cl^*_\lambda(X - cl^*_\lambda(\{\chi_2,\chi_3,\chi_4\})) = cl^*_\lambda(X - \{\chi_2,\chi_3,\chi_4\}) = cl^*_\lambda(\{\chi_1\}) = \{\chi_1,\chi_2,\chi_3\} \supset E$. Therefore E is a strong $\sigma^*-\mathcal{H}$ −open set. But $cl^*_\lambda(int_\lambda(E)) = cl^*_\lambda(int_\lambda(\{\chi_1\})) = cl^*_\lambda(\emptyset) = \emptyset \neq E$. Hence E is not a $\mathcal{H}R$ −closed set.

Theorem 3.9. Let (X,λ) be a GTS with a hereditary class $\mathcal{H}$ and $E \subset X$. If E is a strong $\pi^*-\mathcal{H}$ −open set, then $cl^*_\lambda(E)$ is a strong $\sigma^*-\mathcal{H}$ −open set.

Proof. Let E be a strong $\pi^*-\mathcal{H}$ −open set. Then, $E \subset int^*_\lambda(cl^*_\lambda(E))$, by Lemma $cl^*_\lambda(E) \subset cl^*_\lambda\left(int^*_\lambda(cl^*_\lambda(E))\right) \Longrightarrow cl^*_\lambda(E)$ is a strong $\sigma^*-\mathcal{H}$ −open set.

Theorem 3.10. Let (X,λ) be a GTS with a hereditary class $\mathcal{H}$ and $E \subset X$. If E is a strong $\sigma^*-\mathcal{H}$ −open set, then $int^*_\lambda(E)$ is a strong $\pi^*-\mathcal{H}$ −open set.

Proof. Let E be a strong $\sigma^*-\mathcal{H}$ −open set. Then $E \subset cl^*_\lambda(int^*_\lambda(E))$, by Lemma $int^*_\lambda(E) \subset int^*_\lambda\left(cl^*_\lambda(int^*_\lambda(E))\right) \Longrightarrow int^*_\lambda(E)$ is a $\pi^*-\mathcal{H}$ −open set.

Theorem 3.11. Let (X,λ) be a GTS with a λ −codense hereditary class $\mathcal{H}$ and $E \subset X$. Then E is a $\alpha-\mathcal{H}$ −open set $\Leftrightarrow$ E is a strong $\sigma^*-\mathcal{H}$ −open set and strong $\pi^*-\mathcal{H}$ −open set.

Proof. Let E be a $\alpha-\mathcal{H}$ −open set. Since every $\alpha-\mathcal{H}$ −open set is both a $\sigma-\mathcal{H}$ −open set and $\pi-\mathcal{H}$ −open set. Also, every $\sigma-\mathcal{H}$ −open is a strong $\sigma^*-\mathcal{H}$ −open set and every $\pi-\mathcal{H}$ −open set is a strong $\pi^*-\mathcal{H}$ −open set. Conversely, Suppose E is a strong $\sigma^*-\mathcal{H}$ −open set and strong $\pi^*-\mathcal{H}$ −open set. Then we have, $E \subset int^*_\lambda(cl^*_\lambda(E)) \subset int^*_\lambda\left(cl^*_\lambda\left(cl^*_\lambda(int^*_\lambda(E))\right)\right) = int^*_\lambda\left(cl^*_\lambda(int^*_\lambda(E))\right) = int_\lambda\left(cl^*_\lambda(int_\lambda(E))\right)$. Hence E is $\alpha-\mathcal{H}$ −open set.

Theorem 3.12. Let (X,λ) be a GTS with a hereditary class $\mathcal{H}$ and $E\subset X$ and E is $\lambda-\pi-$open set. If E is a semi closed set, then E is a strong $\sigma^*-\mathcal{H}-$open set.

Proof. Suppose E is a semi closed set. Then, $int_\lambda(cl_\lambda(E))=int_\lambda(E)$. Since E is a $\lambda-\pi-$open set. Then, we have $E\subset int_\lambda(cl_\lambda(E))=int_\lambda(E)\subset int_\lambda(E)\cup int_\lambda^*(E)=cl_\lambda^*(int_\lambda(E))\subset cl_\lambda^*(int_\lambda^*(E))$. Therefore E is a strong $\sigma^*-\mathcal{H}-$open set.

Theorem 3.13. Let (X,λ) be a GTS with $\lambda-$codense hereditary class $\mathcal{H}$ and $E\subset X$ be a λ^*-dense in-itself. If E is a strong $\sigma^*-\mathcal{H}-$open set, then E is a strong $\beta-\mathcal{H}-$open set.

Proof. Let E be a strong $\sigma^*-\mathcal{H}-$open set. Then $E\subset cl_\lambda^*(int_\lambda^*(E))\subset cl_\lambda^*(int_\lambda^*(E^*))=cl_\lambda^*\left(int_\lambda^*(cl_\lambda^*(E))\right)$. *There E is a strong $\beta-\mathcal{H}-$ open set.*

Theorem 3.14. Let (X,λ) be a GTS with a hereditary class $\mathcal{H}$ and $E\subset X$ and E is λ^*-perfect set. If E is almost strong$-\mathcal{H}-$open set, then E is a strong $\sigma^*-\mathcal{H}-$open set.

Proof. Let E is a almost strong $\mathcal{H}-$open set. Then $E\subset cl_\lambda^*(int_\lambda(E^*))$. Since E is λ^*-perfect set, we have $E=E^*\Longrightarrow E\subset cl_\lambda^*(int_\lambda(E^*))=cl_\lambda^*(int_\lambda(E))\subset cl_\lambda^*(int_\lambda^*(E))$. Therefore E is a strong $\sigma^*-\mathcal{H}-$open set.

Theorem 3.15. Let (X,λ) be a GTS with a hereditary class $\mathcal{H}$ and $E\subset X$ and E is λ^*-perfect set. If E is $\mathcal{H}-$open set, then E is a strong $\sigma^*-\mathcal{H}-$open set.

Proof. Let E is a $\mathcal{H}-$open set. Then $E\subset int_\lambda(E^*)$. Since E is a λ^*-perfect set and λ^* is finer the λ, $E\subset int_\lambda(E)\subset cl_\lambda^*(int_\lambda(E))\subset cl_\lambda^*(int_\lambda^*(E))$. Hence E is a strong $\sigma^*-\mathcal{H}-$open set.

Theorem 3.16. Let (X,λ) be a GTS with a hereditary class $\mathcal{H}$ and $E,F\subset X$. If E is a strong $\sigma^*-\mathcal{H}-$open set and F is a $\lambda-\sigma-$ open set, then $E\cup F$ is a $\sigma^*-\mathcal{H}-$open set.

Proof. Let E be a strong $\pi^*-\mathcal{H}-$open set and F be a $\lambda-\sigma-$open set. Then $E\subset cl_\lambda^*(int_\lambda^*(E))$ and $F\subset cl_\lambda(int_\lambda(F))$. Since λ^* finer than λ, now $E\cup F\subset cl_\lambda^*(int_\lambda^*(E))\cup cl_\lambda(int_\lambda(F))\subset cl_\lambda(int_\lambda^*(E))\cup cl_\lambda(int_\lambda^*(F))\subset cl_\lambda\left((int_\lambda^*(E))\cup(int_\lambda^*(F))\right)\subset cl_\lambda(int_\lambda^*(E\cup F))$. Hence $E\cup F$ is a $\sigma^*-\mathcal{H}-$open set.

Theorem 3.17. Let (X,λ) be a QTS with a hereditary class $\mathcal{H}$ and $E,F\subset X$. If E is a strong $\sigma^*-\mathcal{H}-$open set and F is a $\lambda-$open set, then $E\cap F$ is an strong $\sigma^*-\mathcal{H}-$open set.

Proof. Let E be a strong $\sigma^*-\mathcal{H}-$open set and F be a $\lambda-$open set. We have $E\subset cl_\lambda^*(int_\lambda^*(E))$ and $F=int_\lambda(F)$. Now, $E\cap F\subset cl_\lambda^*(int_\lambda^*(E))\cap int_\lambda(F)=\left[int_\lambda^*(E)\cup(int_\lambda^*(E))^*\right]\cap int_\lambda(F)=[int_\lambda^*(E)\cap int_\lambda(F)]\cup\left[(int_\lambda^*(E))^*\cap int_\lambda(F)\right]\subset[int_\lambda^*(E)\cap int_\lambda^*(F)]\cup\left[(int_\lambda^*(E))^*\cap int_\lambda^*(F)\right]\subset[int_\lambda^*(E\cap F)]\cup[int_\lambda^*(E)\cap int_\lambda^*(F)]^*\subset[int_\lambda^*(E\cap F)]\cup[int_\lambda^*(E\cap F)]^*=cl_\lambda^*(int_\lambda^*(E\cap F))$. Therefore, $E\cap F$ is a strong $\sigma^*-\mathcal{H}-$open set.

Theorem 3.18. Let (X,λ) be a GTS with a hereditary class $\mathcal{H}$ and $E\subset X$. Then E is Strong $\sigma^*-\mathcal{H}-$closed$\Leftrightarrow int_\lambda^*(cl_\lambda^*(E))\subset E$.

Proof. Let E be a Strong $\sigma^*-\mathcal{H}-$closed $\Leftrightarrow X-E$ is a strong $\sigma^*-\mathcal{H}-$open set $\Leftrightarrow(X-E)\subset cl_\lambda^*(int_\lambda^*(X-E))=cl_\lambda^*(X-cl_\lambda^*(E))=X-int_\lambda^*(cl_\lambda^*(E))\Leftrightarrow int_\lambda^*(cl_\lambda^*(E))\subset E$.

Theorem 3.19. Let (X,λ) be a GTS with a hereditary class $\mathcal{H}$ and $E,F\subset X$. If E is a strong $\sigma^*-\mathcal{H}-$closed set and F is a weakly semi$-\mathcal{H}-$closed set, then $E\cap F$ is a $\beta^*-\mathcal{H}-$closed set.

Proof. Let E be a strong $\pi^*-\mathcal{H}-$closed set and F be a weakly semi$-\mathcal{H}-$closed set. Then $E\supset int_\lambda^*(cl_\lambda^*(E))$ and $F\supset int_\lambda^*\left(cl_\lambda(int_\lambda(F))\right)$.

Now, $E\cap F\supset int_\lambda^*(cl_\lambda^*(E))\cap int_\lambda^*\left(cl_\lambda(int_\lambda(F))\right)$ by Lemma,

$E\cap F\supset int_\lambda\left(cl_\lambda^*(int_\lambda(E))\right)\cap int_\lambda\left(cl_\lambda^*(int_\lambda(F))\right)=int_\lambda\left(cl_\lambda^*(int_\lambda(E))\cap cl_\lambda^*(int_\lambda(F))\right)\supset int_\lambda\left(cl_\lambda^*(int_\lambda(E)\cap int_\lambda(F))\right)=int_\lambda\left(cl_\lambda^*(int_\lambda(E\cap F))\right)$. Hence $E\cap F$ is a $\beta^*-\mathcal{H}-$closed set.

Theorem 3.20. Let (X,λ) be a GTS with a hereditary class $\mathcal{H}$ and $E,F\subset X$. If E is a strong $\sigma^*-\mathcal{H}-$closed set and F is a $t-\mathcal{H}-$set, then $E\cap F$ is a $\sigma^*-\mathcal{H}-$closed set.

Proof. Let E be a strong $\pi^*-\mathcal{H}-$closed set and F is a $t-\mathcal{H}-$ set. Then $E\supset int_\lambda^*(cl_\lambda^*(E))$ and $int_\lambda(F)=int_\lambda(cl_\lambda^*(F))$. Now, $E\cap F\supset int_\lambda^*(cl_\lambda^*(E))\cap F\supset int_\lambda^*(cl_\lambda^*(E))\cap int_\lambda(F)\supset int_\lambda^*(cl_\lambda^*(E))\cap int_\lambda(cl_\lambda^*(F))\supset int_\lambda(cl_\lambda^*(E))\cap int_\lambda(cl_\lambda^*(F))=int_\lambda(cl_\lambda^*(E)\cap cl_\lambda^*(F))\supset int_\lambda(cl_\lambda^*(E\cap F))$. Therefore E is a $\sigma^*-\mathcal{H}-$closed set.

Theorem 3.21. Let (X,λ) be a GTS with a hereditary class $\mathcal{H}$ and $E\subset X$. Then E is a strong $\sigma^*-\mathcal{H}-$open set$\Leftrightarrow$ there exists a strong $\sigma^*-\mathcal{H}-$open set F such that $int_\lambda^*(F)\subset E\subset F$.

Proof. Let E be a strong $\pi^*-\mathcal{H}-$closed set. Then $E\supset int_\lambda^*(cl_\lambda^*(E))$. Let us assume $F=cl_\lambda^*(E)$ be a λ^*-closed set. There exists a strong $\sigma^*-\mathcal{H}-$open set F such that by Lemma, $int_\lambda^*(F)=int_\lambda^*(cl_\lambda^*(\mathrm{E}))\subset E\subset cl_\lambda^*(E)=F$. Conversely, If F is a strong $\sigma^*-\mathcal{H}-$closed set such that $int_{\lambda}^{*}(F) \subset E \subset F$. Then by Lemma, $int_{\lambda}^{*}(E) = int_{\lambda}^{*}(F)$. Since F is a strong $\sigma^{*}-\mathcal{H}$ −closed set, $\Longrightarrow int_{\lambda}^{*}(cl_{\lambda}^{*}(F)) \subset F$ and hence $E \supset int_{\lambda}^{*}(F) \supset int_{\lambda}^{*}\left(int_{\lambda}^{*}(cl_{\lambda}^{*}(F))\right) = int_{\lambda}^{*}(cl_{\lambda}^{*}(F)) = int_{\lambda}^{*}(cl_{\lambda}^{*}(E))$ it follows that, $E \supset int_{\lambda}^{*}(cl_{\lambda}^{*}(E))$. Therefore E is a strong $\sigma^{*}-\mathcal{H}$ −closed set.

IV. CONCLUSION

As stated in the abstract, we have defined the strong $\sigma^{*}-\mathcal{H}$ −open set in a Generalized Topological space (X,λ) with a hereditary class $\mathcal{H}$. Also, we discuss its properties and give various characterization of this sets. A definition of a strong $\sigma^{*}-\mathcal{H}$ −open set was defined, it has been shown that the concept of a strong $\sigma^{*}-\mathcal{H}$ −open set is weaker than the concept of a$\sigma-\mathcal{H}$ −open set and stronger than the concept of $\sigma^{*}-\mathcal{H}$ −open set. A discussion investigate the relationship between strong $\sigma^{*}-\mathcal{H}$ −open set and some known concepts Generalized Topological space (X,λ) with a hereditary class $\mathcal{H}$. Moreover, many counter examples were established.

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